

An Algebraic Treatment of Totally Linear Partial Differential Equations

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ABSTRACT. We construct the field generated by n algebraically independent elements, and show that the linear space of derivations over this field is faithfully represented by the linear space of the $n - th$ fold Cartesian product of this field acting through inner product on the gradient of this field. We prove also that functional independence of a set in this field is equivalent to linear independence of the gradient set in the vector space of Cartesian product. It is also shown that every subfield S of A which is generated by $(n - 1)$ functionally independent elements defines a one- dimensional space of derivations, such that each member L of the latter subspace has S as its kernel. Each coset of the multiplicative subgroup S defines a non-homogeneous differential operator $L + q$ whose kernel coincides with this coset. We prove also that every element in A defines a coset of the subgroup $Ker(L + q)$ in the additive group A , on which $L + q$ is constant.

1. INTRODUCTION

In a recent work [4] the relations between the solutions of three types of partial differential equations involving first order differential operators were obtained. It was shown that if L is a homogeneous differential operator of class C^0 on a manifold M , q is C^0 function on M then all operators of the form $L + q$ are isomorphic to each other when acting on appropriate Hilbert spaces of functions on M . Neat relations between the solutions of the totally linear partial differential equations (i) $L\phi = 0$, (ii) $(L + q)\psi = 0$, (iii) $(L + q)\chi = b$, where b is a C^0 function, were derived. More precisely, it was shown that if $\eta \in Ker(L + q)$, is any particular solution of (ii) then the set of solutions of (ii) is given by $Ker(L + q) = \eta Ker L$. If ζ is any solution of (iii) then the set of solutions of (iii) is given by $\zeta + Ker(L + q)$. The set of solutions of (i) is a sub-algebra of $C^1(M)$, whereas the set of solutions of (ii), $Ker(L + q)$, is a sub-space of the vector space $C^1(M)$. The algebraic features of these results encourages us to seek an algebraic approach to this problem. Following up this line of thoughts

- we first formulate the problem in a field A generated by a finite number of elements, say n elements.

- study derivations in the algebra defined naturally by this field, and show that the set of derivations is an n -dimensional vector space over the field A . In this concern we come close to the treatment of vector fields in manifold theory [1].
- show that functional independence of a set in the field A is equivalent to linear independence of the gradient of this set in the vector space A^n .
- Utilize the latter result to find the maximum number of functionally independent elements in $\text{Ker}L$.
- Apply the latter results to the solution of equations (ii) and (iii).
- show that every subfield S of A , which is generated by $(n - 1)$ functionally independent elements defines a derivation whose kernel is S , and that every coset of this multiplicative subgroup defines a non-homogeneous differential operator whose kernel is this coset.

2. ALGEBRAIC SETTING

Let A be a field of characteristic 0, and let K be the sub-field generated by the identity element $1 \in A$. The field K is isomorphic to the field of rational number Q , and will be called the scalar field. The field A is a vector space over the scalar field K , and hence it is a commutative algebra (A, K) with an identity 1 and every element in (A, K) , that is different of zero, is invertible.

We assume that the field A is generated by a set $G = \{x_1, \dots, x_n\}$ consisting of n distinct elements that are different from the field's zero, 0, and identity, 1, and that the set G is functionally independent in the sense that there is no conceivable polynomial in the elements of this set equals to zero except the zero polynomial (appendix 1). It follows from definition of a generating set that every element $\psi \in A$ is a function of the elements of the generating set G . A typical element $\psi \in A$ is of the form

$$\psi = P_1 P_2^{-1}, \quad (1)$$

where P_1 and $P_2 \neq 0$ are two arbitrary polynomials in elements of G . More details about the field A , the correlation between algebraic and functional independence, and how to construct a basis in the infinite dimensional vector space (A, K) is given in appendix 1.

We assume also that the field A is partially ordered by a partial ordering relation (\leq) that has the following additional properties

- (i) $\psi_1 > \psi_2, \varphi_1 \geq \varphi_2 \Rightarrow \psi_1 + \varphi_1 > \psi_2 + \varphi_2$.
- (ii) $\psi^2 \geq 0$.

From (ii) and the fact that every element in $A - \{0\}$ is invertible, we get

$$(iii) \quad \psi^2 = 0 \Leftrightarrow \psi = 0.$$

From (i) and (iii), it follows that

$$(iv) \quad \psi^2 + \varphi^2 = 0 \Leftrightarrow \psi = 0, \varphi = 0.$$

Consider the Cartesian product A^n . The set A^n forms an n -dimensional vector space over the field A , when addition and multiplication by a scalar from A are

defined coordinate wise. We shall denote this vector space by (A^n, A) . The mapping $\prec \cdot \mid \cdot \succ: A^n \times A^n \rightarrow A$, defined by

$$\prec (a_1, \dots, a_n) \mid (b_1, \dots, b_n) \succ = \sum_{i=1}^n a_i b_i \quad (2)$$

is left linear, symmetric, and positive definite, and hence is an inner product in A^n . The property of positive definiteness can be checked using the additional property (iv) of the partial ordering relation.

3. THE SPACE OF DERIVATIONS

Let $L: (A, K) \rightarrow (A, A)$ be a linear mapping that satisfy the condition

$$L(\psi\varphi) = \psi(L\varphi) + \varphi(L\psi). \quad (3)$$

In other words L is a derivation [2], [3] from the algebra (A, K) to the algebra (A, A) . It follows from definition that

$$L\alpha = 0 \quad \forall \alpha \in K, \quad L\psi^n = n\psi^{n-1}L\psi \quad \forall n \in \mathbb{Z}. \quad (4)$$

The set of derivations from (A, K) to (A, A) forms a vector space H over the field A , which we denote by (H, A) , and conveniently call the space of homogeneous first order differential operators in (A, K) . Let $B = \{\partial_1, \dots, \partial_n\}$ be a subset of (H, A) satisfying the property

$$\partial_i x_j = \delta_{ij} \quad (i, j = 1, \dots, n). \quad (5)$$

We shall prove that the set B is a basis of the vector space (H, A) , and hence the dimension of the latter space is n . Our proof is quite similar to that followed in manifold theory when proving that the set B is a basis for the tangent space at some point of a manifold [1]. Let $L \in (H, A)$ and assume that $Lx_i = a_i$ ($i = 1, \dots, n$). Now, if $\psi \in A$ is arbitrary then it has the form (1). Acting by L on ψ , we show by a straight forward calculation, that

$$L\psi = a_1\partial_1\psi + \dots + a_n\partial_n\psi. \quad (6)$$

Hence

$$L = a_1\partial_1 + \dots + a_n\partial_n, \quad (7)$$

and B is a generating set in (H, A) . To show that B is linearly independent in the vector space (H, A) , we consider the linear combination

$$\sum_{i=1}^n a_i \partial_i = 0 \quad (a_i \in A) \quad (8)$$

Acting by both sides of the latter equality on x_k we get $a_k = 0$, which is true for all $k \in [1, n]$. Hence the set B is linearly independent, and B is a basis in (H, A) .

Note that every $L \in (H, A)$ is uniquely defined by its values at the elements of the generating set G , and that varying these values in every possible way leads to the set of all possible derivations from (A, K) to (A, A) which are consistent with (5). We note also that one could have set $\partial_i x_j = \delta_{ij} f_i(x)$ to obtain consequently $L = \sum a_i f_i^{-1} \partial_i$. Though this not consistent with the familiar calculus, still it may be interesting to study the unfamiliar derivations and the associated differential equations. However we shall not consider this topic in our present work.

We derive one more rule of differentiation, namely that concerned with the chain differentiation. Let ϕ be a function from A^n to A , where $x = (x_1, \dots, x_n) \rightarrow \phi(x)$. The set of values of all functions $\psi(\phi(x))$ forms a subfield A_ϕ of the field A , and a sub-algebra (A_ϕ, K) of (A, K) . This subfield is generated by a single element ϕ . A typical element of A_ϕ , is of the form $\psi(\phi) = P_m(\phi)(P_l(\phi))^{-1}$, where $P_m(\phi) = \sum_{r=1}^m \alpha_r$

$\phi^r, (\alpha_r \in K)$. It follows that the space of derivation from (A_ϕ, K) to (A_ϕ, A_ϕ) is one dimensional, and hence all derivations on this space have the form $f(\phi)D_\phi$, where $f(\phi) \in A_\phi$, and $D_\phi \phi = 1$.

Though a bit tedious, it is straight forward to check that

$$\partial_i \psi(\phi(x)) = D_\phi \psi(\phi) \partial_i \phi(x)$$

The last relation can be checked by substituting for ϕ the general expression (1) of an element in (A, K) . The last relation shows that

$$(\partial_i \psi)(\partial_j \phi) = (\partial_j \psi)(\partial_i \phi) \quad (i, j = 1, \dots, n),$$

or

$$\text{rank} \begin{bmatrix} \partial_1 \psi & \dots & \partial_n \psi \\ \partial_1 \phi & \dots & \partial_n \phi \end{bmatrix} < 2,$$

which expresses the fact that ψ and ϕ are functionally dependent.

In a similar way the set of functions of the form $\psi = \psi(\phi_1, \dots, \phi_r)$, ($r \leq n$), where $\phi_i(x)$, ($i = 1, \dots, r$) are functionally independent, forms a subfield $A_{1\dots r}$ of A which is generated by $\{\phi_i (i = 1, \dots, r)\}$, and a sub-algebra of the algebra (A, K) . The space of derivations in this sub-algebra is r -dimensional, with $\{D_1, \dots, D_r\}$, where $D_i \phi_j = \delta_{ij}$, as a basis. It can be checked that

$$\partial_i \psi(\phi_1(x), \dots, \phi_n(x)) = (D_1 \psi)(\partial_i \phi_1) + \dots + (D_r \psi)(\partial_i \phi_r). \quad (9)$$

From the latter relation and (7) it follows that

$$L\psi(\phi_1(x), \dots, \phi_r(x)) = (D_1 \psi)(L\phi_1) + \dots + (D_r \psi)(L\phi_r). \quad (10)$$

4. LINK BETWEEN FUNCTIONAL AND LINEAR INDEPENDENCE

Let $\nabla \equiv (\partial_1, \dots, \partial_n) : (A, K) \rightarrow (A^n, A)$ be defined by

$$\nabla \phi = (\partial_1 \phi, \dots, \partial_n \phi) \quad \phi \in A. \quad (11)$$

The operator ∇ , to be called gradient, is linear and satisfy the identity $\nabla(\psi\phi) = \phi(\nabla\psi) + \psi(\nabla\phi)$. Hence ∇ is a derivation from (A, K) to (A^n, A) . If $\Phi = \{\phi_1, \dots, \phi_r\} \subset A$, where $(r \leq n)$, then the vectors $\nabla\phi_1, \dots, \nabla\phi_r \in (A^n, A)$. We shall show that the functional independence of the set of elements Φ in the field A is equivalent to the linear independence of the set vectors $\nabla\Phi = \{\nabla\phi_1, \dots, \nabla\phi_r\}$ in the vector space (A^n, A) . We shall call $\nabla\Phi$ the gradient of the set Φ . Consider the functional relation

$$\psi(\phi_1, \dots, \phi_r) = 0. \quad (12)$$

We have then

$$\sum_{j=1}^r (\partial_i \phi_j)(D_j \psi) = 0 \quad (i = 1, \dots, n). \quad (13)$$

Note first that $D_j \psi = 0 \in (A^r, A)$ is equivalent to $\psi = \alpha$, where $\alpha \in K$. However, by (12), we find that $D_i \psi = 0 \Leftrightarrow \psi = 0$. i.e. $D_i \psi = 0$ is equivalent to the functional independence of the set Φ . Equation (13) asserts that the r -vector $(D_j \psi) \in (A^r, A)$ is orthogonal to the r -vectors $\partial_i \phi_j$ ($i = 1, \dots, n$). If the r -vector $(D_j \psi)$ is not zero then it can be orthogonal to $(r - 1)$ vectors in (A^r, A) at most. Hence the rank of the $(n \times r)$ matrix $(\partial_i \phi_j)$ is less or equal to $(r - 1)$, and the set of vectors $\{\nabla\phi_j : j = 1, \dots, r\} \subset (A^n, A)$ is linearly dependent. Conversely, and if the set of vectors $\nabla\phi_j \in (A^n, A)$, ($j = 1, \dots, r$) is linearly dependent then the rank of the matrix $(\partial_i \phi_j)$ is less than r . As a result equation (13) can have a non-trivial solution $(D_j \psi) \neq 0$. Therefore the set of functions Φ is functionally dependent.

It is clear that if $r > n$, then the set of functions Φ is certainly functionally dependent. This is because the system of linear equations (13) is under determined, and has consequently a non-trivial solution $(D_1 \psi, \dots, D_r \psi)$. We state the latter results in the following theorem

Proposition 1. 1. A subset $\Phi = \{\phi_1, \dots, \phi_r\}$ of the field A is functionally independent if and only if the set of vectors $\nabla\Phi = \{\nabla\phi_1, \dots, \nabla\phi_r\} \subset (A^n, A)$ is linearly independent. In other words, the set Φ is functionally independent iff its gradient is linearly independent.

2. If $r > n$ then Φ is functionally dependent. Hence there can exists in A no more than n functionally independent elements. If $r \leq n$ then Φ is functionally independent if and only if

$$\text{rank}(\partial_i \phi_j) = r \quad (i \in [1, n], j \in [1, r]) \quad (14)$$

5. THE HOMOGENEOUS LINEAR EQUATION $L\phi = 0$.

Let $L = \sum_{i=1}^n a_i \partial_i \in (H, A)$. The correspondence

$$L \in (H, A) \leftrightarrow (a_1, \dots, a_n) \in (A^n, A) \quad (15)$$

is a vector space isomorphism. Consider a subset ∇A of A^n , defined by

$$\nabla A = \{\nabla \phi : \phi \in A\}. \quad (16)$$

The set ∇A will be called the gradient of the field A . It is clear that ∇A is a subspace of the vector space (A^n, K) . Since

$$L\phi = \langle (a_1, \dots, a_n) \mid \nabla \phi \rangle$$

we may identify L by the operator

$$\langle (a_1, \dots, a_n) \mid \rangle : (\nabla A, K) \rightarrow A \quad (17)$$

If ϕ_k is a solution of the homogeneous differential equation (*HDE* for short)

$$L\phi = 0, \quad (18)$$

then

$$\langle (a_1, \dots, a_n) \mid \nabla \phi_k \rangle = 0. \quad (19)$$

It follows that $L\phi_k = 0$ if and only if $\nabla \phi_k \in L^\perp$, where $L^\perp \subset (A^n, A)$ is the set of all vectors in (A^n, A) which are orthogonal to the vector $(a_1, \dots, a_n) \in (A^n, A)$. The set L^\perp is clearly an $(n-1)$ -dimensional subspace of the vector space (A^n, A) . Since $L\phi_k = 0 \Leftrightarrow \phi_k \in \ker L$, we have $\nabla \ker L \subset L^\perp$. Thus $\nabla \ker L$ can have at most $(n-1)$ elements, which are linearly independent, and hence $\ker L$ can have at most $(n-1)$ elements that are functionally independent. Therefore the *HDE* (18) can have at most $(n-1)$ functionally independent solutions, which is a well-known result in partial differential equations.

6. NON-HOMOGENEOUS OPERATORS

Let $q \in A$, and consider the operator $L + q : (A, K) \rightarrow (A, A)$, defined by

$$(L + q)\psi = L\psi + q\psi. \quad (20)$$

This operator is linear. The set of such operators $NH = \{L + q : L \in H, q \in A\}$ form a vector space over the field A . Through the inclusion mapping $L \in (H, A) \rightarrow (L + 0) \in (NH, A)$, where $0 \in A$, we may write $(H, A) \subset (NH, A)$. We shall show that the set $NB = \{\partial_1, \dots, \partial_n, 1\} \subset (NH, A)$ is a basis in (NH, A) , and thus (NH, A)

is an $(n + 1)$ -dimensional vector space. It clear first that NB generates the vector space (NH, A) . Consider next the linear combination

$$\sum_{i=1}^n a_i \partial_i + b.1 = 0$$

Acting by both sides on $\alpha \in K$ we get $b\alpha = 0$ for every α , and hence $b = 0$. Acting on x_k ($k = 1, \dots, n$) we get $a_k = 0$ ($k = 1, \dots, n$). Therefore the set NB is linearly independent, and hence is a basis of (NH, A) .

An alternative way of reaching the latter results is based on identifying (NH, A) with $(H, A) \times (A, A)$ through the isomorphism $L + q \in (NH, A) \longleftrightarrow (L, q) \in (H, A) \times (A, A)$.

If $L = \sum a_i \partial_i$ then the correspondence

$$L + q \in (NH, A) \longleftrightarrow (a_1, \dots, a_n, q) \in (A^{n+1}, A) \quad (21)$$

is a vector space isomorphism. Since

$$(L + q)\psi = \langle (a_1, \dots, a_n, q) \mid (\nabla \psi, \psi) \rangle \in A, \quad (22)$$

we may identify $L + q$ by the operator

$$\langle (a, \dots, a, q) \mid (\cdot, \cdot) \rangle : (\nabla A \times A) \rightarrow A. \quad (23)$$

Now $\psi \in A$ is a solution of the non-homogeneous differential equation ($NHDE$ for short)

$$(L + q)\psi = 0, \quad (24)$$

if and only if

$$\langle (a_1, \dots, a_n, q) \mid (\nabla \psi, \psi) \rangle = 0.$$

Therefore the set of solutions of the $NHDE$ (24), namely $Ker(L + q)$ satisfy the condition

$$\nabla Ker(L + q) \times Ker(L + q) \subset (L + q)^\perp, \quad (25)$$

where $(L + q)^\perp$ is the set of all vectors in (A^{n+1}, A) which are orthogonal to the vector $(a_1, \dots, a_n, q) \in (A^{n+1}, A)$. Since the set $(L + q)^\perp$ is an n -dimensional sub-space of the vector space (A^{n+1}, A) , we conclude that there exist at most n linearly independent vectors $(\nabla \psi_k, \psi_k) \in \nabla \ker L \times \ker L$ ($k = 1, \dots, n$). Equivalently there exist at most n solutions of the $NHDE$ (24) such that

$$rank(\partial_i \psi_k, \psi_k) = n. \quad (26)$$

When the last equation holds, it is satisfied in particular, by n solutions ψ_k such that $rank(\partial_i \psi_k) = n$.

7. LINK BETWEEN THE SOLUTIONS OF *HDEs* AND *NHDEs*

The set of solutions of the *HDE* (18), namely $\ker L$, is a sub-algebra of the algebra (A, K) . For, $\ker L$ is a subspace of the vector space (A, K) because L is linear, and $\ker L$ is closed under multiplication because L is a derivation. The set of solutions of the *NHDE* (24) is a subspace of the linear space (A, K) . In what follows we shall assume that $\eta \in A$ is any particular solution of the *NHDE* (24), i.e. $\eta \in \ker(L + q)$.

Proposition 2. *The set of solutions of the NHDE (24) is given by*

$$\ker(L + q) = \eta \ker L. \quad (27)$$

Proof: Let $\phi \in \ker L$, then

$$(L + q)(\eta\phi) = \phi(L + q)\eta + \eta L\phi = 0.$$

Therefore $\eta\phi \in \ker(L + q)$, and $\eta \ker L \subseteq \ker(L + q)$. Let $\psi \in \ker(L + q)$. By equation (24) which is satisfied by ψ and η , we have $\psi L\eta = \eta L\psi$. Now

$$L(\psi\eta^{-1}) = \eta^{-2}(\eta L\psi - \psi L\eta) = 0.$$

Hence $\psi\eta^{-1} \in \ker L$, and $\psi = \eta(\psi\eta^{-1}) \in \eta \ker L$, and the theorem is proved.

Corollary 3. *In the context of the latter proposition we proved that if ξ and η are particular solutions of the NHDE (24) then $\xi\eta^{-1}$ is a particular solution of the HDE (18). However this result is contained in equation (27) which gives $\xi\eta^{-1}\ker L = \ker L$. But since $\ker L$ is a subgroup of the multiplicative group A , the last equation implies that $\xi\eta^{-1} \in \ker L$.*

Corollary 4. *Equation (27) shows that the general solution of (24) is given by the product of a particular solution of this equation by the general solution of equation (18).*

Proposition 5. $\eta \in \ker(L + q) \Leftrightarrow L + q = \eta L\eta^{-1}$.

Proof: It is understood that the latter equation is an operator equation, in which $\eta L\eta^{-1}$ is the composite of three mapping in (A, K) , so that $(\eta L\eta^{-1})\psi = \eta(L(\eta^{-1}\psi))$.

(\Rightarrow) Assume first that $(L + q)\eta = 0$, so that $q = -\eta^{-1}(L\eta)$. Hence

$$\eta L\eta^{-1} = L - \eta\eta^{-2}(L\eta) = L + q.$$

(\Leftarrow) $(L + q)\eta = (\eta L\eta^{-1})\eta = \eta(L1) = 0$. Hence $\eta \in \ker(L + q)$, and the proof is complete..

Consider now the *NHDES* (S is short for (with second side))

$$(L + q)\chi = b, \quad (28)$$

where $b \in A$. If χ_0 is a particular solution of this equation and χ is any other solution, then

$$(L + q)(\chi - \chi_0) = 0, \quad (29)$$

and the solution of the *NHDES* is reduced to the solution of the *NHDE*. Therefore the set of solution O of (28) is given by

$$O = \chi_0 + \text{Ker}(L + q) = \chi_0 + \eta \text{Ker} L. \quad (30)$$

The last relation reads that the set of solutions of a *NHDES* is a coset of the subspace $\text{Ker}(L + q)$ determined by a particular solution χ_0 .

8. CORRESPONDENCE BETWEEN COSETS AND DIFFERENTIAL OPERATORS

Let $\Phi = \{\phi_1, \dots, \phi_{n-1}\}$ be a functionally independent subset in A , and let S be the sub-field generated by Φ . Since $1 \in S$, the scalar field K is a sub-field of S . Evidently, S is a sub-algebra of A . Let $\Psi = \{\psi_1, \dots, \psi_{n-1}\}$ be any other functionally independent subset of S . Since Φ generates S we have

$$\psi_i(x) = f_i(\phi_1(x), \dots, \phi_n(x)) \quad (i = 1, \dots, n-1).$$

Hence

$$\begin{bmatrix} \nabla \psi_1 \\ \dots \\ \nabla \psi_{n-1} \end{bmatrix} = \begin{bmatrix} D_1 f_1 & \dots & D_{n-1} f_1 \\ \dots & \dots & \dots \\ D_1 f_{n-1} & \dots & D_{n-1} f_{n-1} \end{bmatrix} \begin{bmatrix} \nabla \phi_1 \\ \dots \\ \nabla \phi_{n-1} \end{bmatrix}.$$

This relation shows that every vector $\nabla \psi_i (i = 1, \dots, n-1)$ is a linear combination in the elements of the set $\nabla \Phi$. Since $\nabla \Phi$ and $\nabla \Psi$ are both linearly independent in (A^n, A) , it follows that the $(n-1)$ -dimensional vector space generated by $\nabla \Phi$, denoted by $[\nabla \Phi]$ is identical to $[\nabla \Psi]$. We shall denote this sub-space by L^\perp . The orthogonal complement of L^\perp is an one-dimensional space spanned by one vector (a_1, \dots, a_n) . This determines an one dimensional space of homogeneous differential operators spanned by $L = \sum_{i=1}^n a_i \partial_i$ and such that $\text{ker} L = S$. It is clear of course that L is any member in this one dimensional space, and hence L is determined up to a multiplicative function $f(x) \in A$. We conclude therefore that every functionally independent subset of $(n-1)$ elements determines on one hand a sub-field S of A , and on the other hand an one dimensional space of homogeneous differential operators, such that if L is any member of the latter sub-space then $\text{Ker} L = S$.

Define now an equivalence relation on A by

$$\xi \sim \eta \Leftrightarrow \xi \eta^{-1} \in S$$

This relation partitions A to the subgroup S (subgroup with respect to multiplication in A) and all its cosets. Consider the coset defined by η , where $\eta \notin S$. Since $(\eta L \eta^{-1})\eta = 0$, it follows that $\eta \in \ker(L + q)$, where $q = -\eta^{-1}(L\eta)$. If ξ is any other element in the coset ηS , then $(\eta L \eta^{-1})\xi = \eta L(\eta^{-1}\xi) = 0$, because $\eta^{-1}\xi \in S = \ker L$. Therefore every coset ηS of S defines, up to a multiplicative element from A , a unique non-homogeneous operator $L + q$ such that $\eta S = \ker(L + q)$.

Now every proper coset $\eta \ker L = \ker(L + q)$ of the subgroup $\ker L$ is a subgroup of the additive group A . Define on A an equivalence relation by

$$\chi \simeq \chi_0 \Leftrightarrow \chi - \chi_0 \in \ker(L + q).$$

This equivalence relation partitions A into equivalence classes consisting of $\ker(L + q)$ and all its cosets. Take $\chi_0 \in A$, such that $\chi_0 \notin \ker(L + q)$, and set $(L + q)\chi_0 = b$. Let $\chi \in \chi_0 + \ker(L + q)$ be arbitrary. Since $\chi - \chi_0 \in \ker(L + q)$, we have $(L + q)(\chi - \chi_0) = 0$, and hence $(L + q)\chi = (L + q)\chi_0 = b$. Therefore, given any operator $(L + q)$, every $\chi_0 \in A$, determines a coset of $\ker(L + q)$, in the additive group A , on which the operator $L + q$ is constant.

9. APPENDIX 1.

Here we list the ordered set B_P consisting of all one term polynomials (or the non fractional elements) in A according to raising powers

$$\begin{aligned} &1; \\ &x_1, \dots, x_n; \\ &x_1^2, x_1 x_2, \dots, x_1 x_n; x_2^2, x_2 x_3, \dots, x_2 x_n; x_3^2, x_3 x_4, \dots, x_3 x_n; \dots; x_{n-1}^2, x_{n-1} x_n; x_n^2; \\ &x_1^3, x_1^2 x_2, \dots, x_2^3, x_2^2 x_3, \dots \\ &\dots \end{aligned}$$

Now every polynomial $P \in A$ can be written as a linear combination in the elements of B_P . The assumption that the generating set is functionally independent is equivalent to say that the only vanishing linear combination of the elements of B_P is the trivial linear combination. From this property it follows that every polynomial in A can be expressed in a unique way as a linear combination of the elements of B_P . Thus to every polynomial $P \in A$ there corresponds a unique sequence (s_i) in Q , defined by $s_i =$ the i -th coefficient of P on B_P . It clear that the set of all polynomial in A , denoted by $P(A)$ forms an infinite dimensional vector space over the field K , with B_P , as a basis. The vector space $P(A)$ is isomorphic to the set $S(Q)$ consisting of all infinite sequences of rational numbers which have only a finite number of non-zero terms. It is understood of course that addition in $S(Q)$ and multiplication by a scalar from Q is defined term-wise. As an example, there corresponds to the polynomial $2^{-1} + y + 3.4^{-1}x^2 + y^2 + x^2y$, when $n = 2$, the sequence $(1/2, 0, 1, 3/4, 0, 1, 0, 1, 0, 0, \dots)$.

We mention that the general form of a polynomial of degree k in the elements of G is

$$P = \sum_{s=0}^k \sum_{i_1 \dots i_n} \alpha_{i_1 \dots i_n}^{j_1 \dots j_n} x_{i_1}^{j_1} \dots x_{i_n}^{j_n},$$

where the indices are subject to the conditions

$$i_1 < \dots < i_n \leq n, \quad j_1 \geq \dots \geq j_n \geq 0, \quad s = j_1 + \dots + j_n \in [0, k],$$

and where the α' 's are scalars from K . Note that any function of elements of A is a element of A and hence will have the form (1). If this function is zero then the numerator is zero, and therefore functional independence in A is equivalent to algebraic independence. The set $P(A)$ is an integral domain with respect to the usual operations of addition and multiplication, and it is clear that the field A is the quotient field of the integral domain $P(A)$. The set of sequences $S(Q)$ forms also an integral domain, with respect to the usual addition and multiplication of infinite sequences, which is isomorphic to the integral domain $P(A)$. Therefore the quotient field of $S(Q)$ is isomorphic to the quotient field of $P(A)$, which is A .

In order to establish a basis for the vector space (A, K) , we agree on the following: we represent every element $\psi = P_1 P_2^{-1}$ by a pair $(P_1, P_2) \in P(A) \times P(A)^*$, such that the coefficient of the leading term in P_2 is equal to +1, and such that there is no common factor in P_1 and P_2 . Under this convention one can see that (B_P, B_P) is a basis of (A, K) . It is evident that the addition and multiplication in A are defined by $(P_1, P_2) + (p_1, p_2) = (p_2 P_1 + p_1 P_2, p_1 p_2)$, $(P_1, P_2) \cdot (p_1, p_2) = (P_1 p_1, P_2 p_2)$. If we adopt in $S(Q) \times S(Q)^*$ a similar agreement to that adopted in A , we find that A is isomorphic to $S(Q) \times S(Q)^*$.

An alternative to the content of this appendix is to consider each element $\psi = P_1 P_2^{-1}$ as a class of equivalence in $P(A) \times P(A)^*$, in which $(P_1, P_2) \sim (P_3, P_4)$ iff $P_1 P_3 = P_2 P_4$, and to define a similar equivalence relation on $S(Q) \times S(Q)^*$, so that the quotient field of $S(Q)$ is isomorphic to A .

The algebraic notions used in this appendix can be found in [5]

10. GRATITUDE

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